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Analysis of Variational Methods in Image Restoration

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Abstract— In this paper, image restoration using Variational methods based on Partial Differential Equation is presented for the removal of blur and noise. The algorithms presented are aimed to reduce the total variation in the image degraded by the noisy environment. Compared to linear filtering methods which introduce spurious oscillations at the abrupt jumps of the signal, the methods presented found to be superior. Extensive simulations have been carried out to show the effectiveness of our algorithms. Simulation results show improvements both visually and quantitatively compared to the observed image. In addition, it is capable of removing both multiplicative and additive noise from the image that is affected by blur.

Index Terms— additive noise, blur, linear filtering, multiplicative noise, Partial Differential Equation (PDE), Total Variation (TV), image restoration.

1 Introduction

IMAGE Restoration including image denoising and deblurring is one of the fundamental and important research area in image processing. Image restoration is a process of recovering the original image from the image degraded by the noisy environment. Degradation occurs due to the presence of noise and blur in the environment. Random noise is different during acquisition, transmission and processing of images [1]. All deterministic models for degradation are based on the point spread function. The spatial degradation occurs due to atmospheric turbulence, motion blur, and defocusing system [2]. Image restoration is also referred to as image de-blurring or image de-convolution, finds application in the fields such as astronomy, remote sensing, medical image restoration, digital imaging systems [3]-[5] etc.

Partial Differential Equations (PDEs) are widely used to solve the inverse problems. This can be achieved by discretizing such equations which results with a finite number of unknowns. There are several different ways to discretize a PDE. The simplest method uses finite difference approximations to obtain first and second order derivatives. The main advantage of using PDE is simple to implement. The Finite Element Method makes use of the function which contains some degree of smoothness over the entire region.The solution of PDE equations are solved by using Euler Lagrange equations for image processing [6].These techniques finds its application in image inpainting, water marking and image segmentation [7]. Most of the restoration process is ill-posed.

The linear inverse problem that follows $hx = y$ where $h \in R^{n \times m}$ is the blur PSF with $n < m$ is generally underdetermined and it does not have a stable unique solution. The problem is to estimate $x = h^{-1}y$ from the given measurement y. *'h'* is not invertible or a singular matrix.

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In general, linear models work efficiently for denoising which is achieved by low pass filtering the signal. Since both noise and edges contain high frequencies, low pass filtering of piece wise constant signals (PWC) signals introduces spurious oscillations at the edges referred to as Gibb's phenomena [8].So, conventional least square filters cannot be applicable for effective noise removal. For this reason, several non linear methods used by restoration techniques such as wiener filter, median filter, wavelet thresholding has been developed [9] which preserves edges in an efficient manner compared to linear models. One of the simpler and popular Total Variation (TV) method and anisotropic diffusion will provide a well posed solution to such inverse problems by formulating the efficient optimization problems.

The organisation of the paper is summarized as follows.PDE based image restoration is explained in the section 2. Section 3 presents the simulation of the presented algorithms. Section 4 presents the conclusion.

2 PDE based Image Restoration

Total Variation (TV) based image restoration has been introduced by Rudin, Osher, and Fatemi in 1992 [10]. Popular model for nonlinear image de-noising and deblurring is the Total Variation (TV) model. Total Variation determines the overall variation between the signal values.TV is an energy based algorithm. High and low value of the energy function corresponds to noise and clear (without noise) image respectively. By making use of iterative procedure, TV method makes the image to have the lowest energy.

Thus, designing a good energy function results in an efficient image restoration. TV denoising is used as an effective filtering method for recovering piecewise-constant signals [10]. It can be applied to 1D signal by using min-max property and Majorization and Minimization algorithm. It can also be applied to 2D images by the formulation of energy function using Partial differential equations (PDE).The idea is to minimize the variation of the image so that the image becomes more clearer than the noisy image. Total Variation makes use of $l1$ norm of the gradient whereas least square criteria uses $l2$ norm. The development of fast, robust algorithms for TV and related non-linear filtering has been an active research topic.

The concept of heat equation or diffusion is not only applicable for heat transfer. It can be applied to image processing tasks such as image smoothing, inpainting, restoration. The principle behind diffusion process is given by Fick's law. It can be mathematically formulated [11] as

$$J = -D\nabla u \tag{1}$$

This equation states that a concentration gradient ∇u creates a flux J which tries to compensate for this gradient. ∇u and J are related by the diffusion tensor D, a positive definite symmetric matrix. Diffusion is anisotropic, if J and ∇u are not parallel.The continuity equation describes that diffusion makes the energy to move at time't' rather than creating or destroying it. Diffusion equation is expressed as

$$\partial_t u = div(D\nabla u) \tag{2}$$

Image affected by noise have different intensities at each pixel location. This has to be equilibrated by using diffusion process.If the diffusion tensor is a function of the gradient of an evolving image, it results in nonlinear diffusion filters. Linear diffusion is very effective in smoothing but edges are not preserved. Due to the use of Gaussian filter, smoothing has done to remove noise but edges are also get smoothened and yields the blurred image. Linear diffusion filtering does not locate edges properly when moving from finer to coarser scales. So structures which are identified at a coarse scale do not give the right location and have to be traced back to the original image called corresponding problem [12].

2.1 Tikhnov Model

The tikhnov model problem is formulated by the energy function referred to as the objective or cost function is expressed as

$$\min\{E(u)\} = \frac{1}{2}\int_{\Omega}\left|\nabla u(x,y)\right|^2 + \frac{1}{2\lambda}\int_{\Omega}(h*f-u)^2 \tag{3}$$

where $u(x,y)$ is the restored image, f represents the observed image, h refers to the blur operator, the first term represents the prior and the second one represent the data fidelity term. Fig.1 shows the block diagram of image restoration using Tikhnov model.

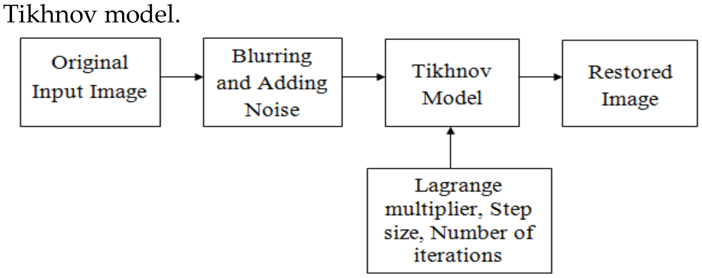

Fig. 1. Block diagram of Tikhnov model

The optimal value of u which minimizes the objective function is obtained by using Newton's method. The equation can be solved by using Euler Lagrange equation, as a gradient of the functional E and represented by ∇E .

$$\frac{\partial u}{\partial t} = -\nabla E = (u_{xx} + u_{yy}) - \frac{1}{\lambda}(f - h*u) \tag{4}$$

This is a kind of heat equation, converges on a steady state solution of 'u'. Using finite difference scheme, the above mentioned PDE can be solved. Algorithm for denoising and deblurring using Tikhnov model can be summarized as follows.

Algorithm for Tikhnov Model

1. Obtain the input image.
2. Blur the input image by using blur kernel of 15 x 15 filter-with values $h(i,j) = \frac{1}{i^2 + j^2 + 1}$ for $7 \le i,j \le 7$.
3. Add random noise to the blurred image standard deviation, σ=2.
4. Set the values for regularization or Lagrange multiplier λ, Number of iterations and step size, dt.
5. Pass the degraded image into Tikhnov model filter to restore the Original image.
6. Determine the first and second order derivatives to update the equation by solving (4).
7. If the number of iteration gets over, display the restored image and analyze it with the PSNR evaluation criterion.

2.2 ROF Model

In Tikhnov Model, the image prior which is considered as a set of smooth image has low gradient value .As a result, it yields a blurred image. Rudin, Osher and Fatemi introduced the ROF model where they introduced the prior term as the ℓ_1 norm. It is expressed by unconstrained or Lagrangian model as

$$\min\{E(u)\} = \int_{\Omega}\left|\nabla u(x,y)\right|d\Omega + \frac{1}{2\lambda}\int_{\Omega}(f-h*u)^2 d\Omega \tag{5}$$

where λ is the Lagrange Multiplier. Fig. 2 shows the block diagram of image restoration using ROF model.

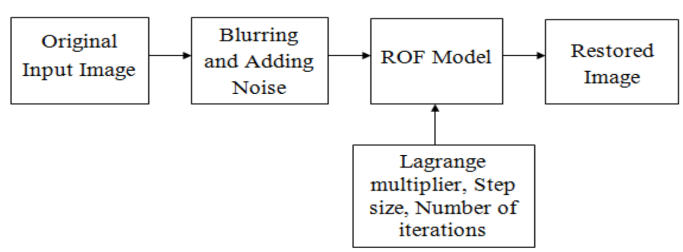

Fig. 2. Block diagram of ROF model

Using gradient descent method, the update equation is expressed in (6).

$$\frac{\partial u}{\partial t} = -\nabla E = \frac{h*(h*u-f)}{\lambda} + \frac{u_{xx}u_y^2 - u_x u_y u_{xy}}{(u_x{}^2 + u_y{}^2)^{\frac{3}{2}}} + \frac{u_{yy}u_x^2 - u_x u_y u_{xy}}{(u_x{}^2 + u_y{}^2)^{\frac{3}{2}}} \tag{6}$$

Algorithm for denoising and deblurring using ROF model can be summarized as follows.

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Algorithm for ROF Model

1. Obtain the input image.
2. Blur the input image by using blur kernel of 15 x 15 filter with values $h(i,j)=\dfrac{1}{i^2+j^2+1}$ for $7 \le i,j \le 7$.
3. Add random noise to the blurred image with standard deviation, σ=2.
4. Set the values for regularization or Lagrange multiplier λ, smudge factor, Number of iterations and step size, dt.
5. Pass the degraded image into ROF model filter to restore the Original image.
6. Determine the first and second order derivatives to update the equation by (6).
7. If the number of iteration gets over, display the restored image and analyze it with the PSNR evaluation criterion.

2.3 Perona Malik Diffusion

A non linear PDE formulation for image smoothing has been introduced by Perona and Malik [13]. Fig.3 shows the block diagram of image restoration using Perona Malik Diffusion model.

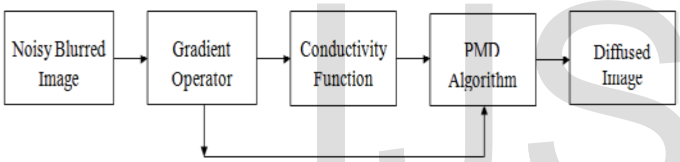


Fig. 3. Block diagram of Perona Malik Diffusion model.

Algorithm for Perona Malik Diffusion (PMD)

1. Obtain the input image.
2. Blur the input image by the Gaussian blur with standard deviation, σ=20 of size 3×3.
3. Add Speckle noise to the blurred image with variance, $\sigma^2 = 0.01$.
4. Determine the gradient of the blurred noisy image
5. Determine the conductivity function, g using the equation (8).
6. Conductivity and the Gradient value of an image are given as the input to the Perona Malik diffusion filter.
7. Obtain the diffused image and evaluate the output by using PSNR, Equivalent Number of Loops (ENL) and Edge Preserving Index (EPI) criterion.

PMD is a non linear filter used for avoiding blurring and localization problems by performing inhomogeneous process at the presence of edges. The diffusion process is position dependent expressed in the equation (7).

$$\frac{\partial I}{\partial t}=\nabla.\left(g(x,y)\nabla I\right) \tag{7}$$

$$g\left(\left\|\nabla I\right\|\right)=\exp\left(\frac{-\left\|\nabla I\right\|^2}{k^2}\right) \tag{8}$$

where $g(x,y)$ is the conductivity function depends on the gradient $\left\|\nabla I\right\|$ of the image at location (x,y), $I(x,y)$ denotes the intensity of the image *I*, K is a Constant parameter controls conductivity function.

In terms of spatial derivatives, diffusion can be written as

$$\frac{\partial I}{\partial t}=\frac{\partial}{\partial x}\left(g(x,y)\frac{\partial I}{\partial t}\right)+\frac{\partial}{\partial y}\left(g(x,y)\frac{\partial I}{\partial t}\right) \tag{9}$$

$$I_{i,j}^{t+dt}=I_{i,j}^{t}+\frac{dt}{2}\begin{bmatrix}(g_{i+1,j}+g_{i,j})(I_{i+1,j}-I_{i,j})\\-(g_{i,j-1}+g_{i,j})(I_{i,j}-I_{i-1,j})\\+(g_{i,j+1}+g_{i,j})(I_{i,j+1}-I_{i,j})\\-(g_{i,j-1}+g_{i,j})(I_{i,j}-I_{i,j-1})\end{bmatrix} \tag{10}$$

Solving equation (9), the update equation obtained is given in the equation above. Perona Malik Diffusion algorithm for denoising and deblurring can be summarized as follows.

2.4 Edge Enhancing Diffusion

PMD is position dependent but not directed. Edge enhancing diffusion has been developed by Weickert to find the orientation of the edges.

$$D=R^T\begin{pmatrix}c1 & 0\\0 & c2\end{pmatrix}R \tag{11}$$

$$R=\frac{1}{\sqrt{(I_u^x)^2+(I_u^y)^2}}\begin{pmatrix}I_x^u & -I_y^u\\I_y^u & I_x^u\end{pmatrix} \tag{12}$$

where the diffusion tensor D is formulated as where *R* is the rotation matrix, defines system aligned with the gradient vector at scale *u*. I_u is the image observed at scale *u*. Fig.4 shows the block diagram of image restoration using Edge Enhancing Diffusion model.

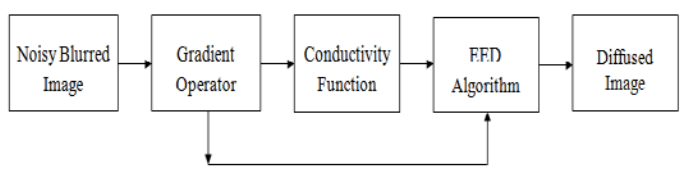


Fig. 4. Block diagram of Edge Enhancing Diffusion model.

The diffusion is directed towards 90 degree to the gradient. This diffusion tensor, D is given in the equation (13).

$$D=\frac{1}{\sqrt{(I_u^x)^2+(I_u^y)^2}}\begin{pmatrix}c_1(I_x^u)^2+c_2(I_y^u)^2 & (c_1-c_2)I_x^uI_y^u\\(c_1-c_2)I_x^uI_y^u & (I_x^u)^2+(I_x^u)^2\end{pmatrix} \tag{13}$$

$$c_1=\frac{1}{5}c_2 \tag{14}$$

$$c_2=\exp\left(\frac{-I_w^u}{k^2}\right) \tag{15}$$

where c_1 and c_2 is the conductivity in the direction of gradient and curves to preserve edges and corners present in the image respectively. Thus, the conductivity along edge is designed to be one fifth of the conductivity across edge. Edge Enhancing Diffusion algorithm for denoising and deblurring can be summarized as follows.

Algorithm for Edge Enhancing Diffusion (EED)

1. Obtain the input image
2. Blur the input image by the Gaussian blur with standard deviation, σ=20 of size 3×3.
3. Add Speckle noise to the blurred image with variance, $\sigma^2 = 0.01$.
4. Determine the gradient of the blurred noisy image.
5. Determine the conductivity functions, c_1 and c_2 using the equations (14) and (15).
6. Conductivity and the Gradient value of an image are given as the input to the Edge Enhancing Diffusion filter.
7. Obtain the diffused image and evaluate the output by using PSNR, Equivalent Number of Loops (ENL) and Edge Preserving Index (EPI) criterion.

3 Simulation Results

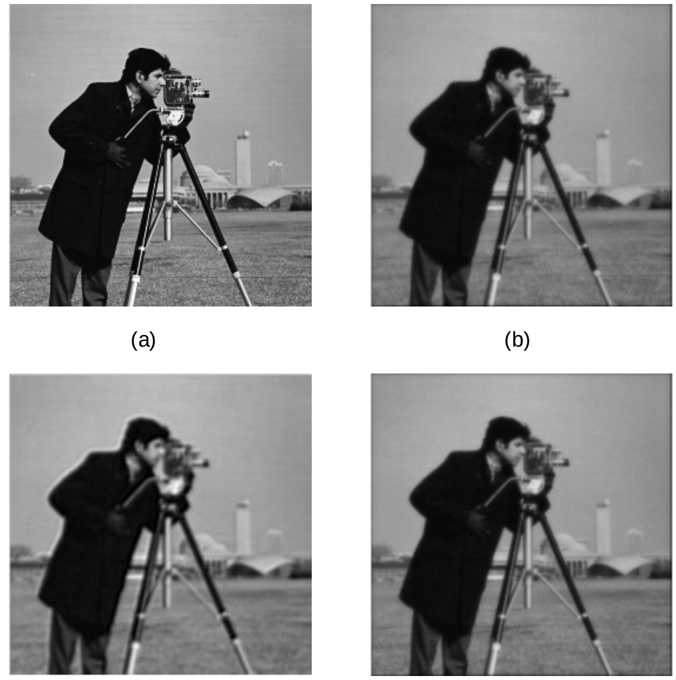
Simulation results of the image restoration using Tikhnov and ROF model is shown in the Fig. 5. Fig. 5(a) shows the input Cameraman image with size 256×256. The input image is blurred with the kernel of 15×15 filter with values $h(i, j) = \frac{1}{i^2 + j^2 + 1}$ for $7 \leq i, j \leq 7$, which is normalised to have a unit sum shown in the Fig. 5(b). Other simulation parameters includes Number of Iterations= 100, Step size, dt = 0.24 and Regularization parameter= 10.

TABLE 1

Comparison of Total Variation Method Algorithms

Images	PSNR (dB)	
	Tikhnov Model	*ROF Model*
Cameraman	25	30
Lena	24	28
Onion	27	33

Fig. 5(c) is the restored image by deblurring and denoising the corrupted image using Tikhnov model with PSNR= 25dB. Fig. 5(d) is the restored image by deblurring and denoising the corrupted image. Since, the image prior designed in Tikhnov model contains low gradient (ℓ_1 norm), edges are smoothened out. Unlike Tikhnov model, ROF model preserves edgesbecause ℓ_1 norm is used to model the prior. From Table 1, it is clear that ROF model shows better performance in image restoration with PSNR of 5 dB greater than that of Tikhnov model. Simulation results of the image restoration using Edge Enhancing Diffusion and Perona Malik Diffusion are shown in the Fig. 6.



(a) (b) (c) (d)

Fig. 5.(a) Original image, Restored image using (b)Tikhnov Model, (c)ROF Model.

Fig. 6(a) shows the input Cameraman image of size 256×256. Fig. 6(b) shows the blurred noisy image is obtained by blurring the input image using Gaussian blur with σ= 20 and including Speckle noise which is multiplicative in nature with σ^2 = 0.04. Other simulation parameters includes Number of Iterations= 100, Step size, dt =0.24, Constant parameter controls conductivity function, K=60.

TABLE 2

Comparison of PMD and EED Algorithms

Images	PMD			EED		
	PSNR (dB)	*ENL*	*EPI*	*PSNR (dB)*	*ENL*	*EPI*
Cameraman	27	128	0.10	29	138	0.20
Lena	30	118	0.07	33	126	0.16
House	29	112	0.09	31	120	0.17

Fig. 6(c) is the diffused image by deblurring and denoising the corrupted image with PSNR= 27dB, ENL= 128, EPI= 0.10 using Perona Malik Diffusion method. Fig. 6(d) is the diffused image by deblurring and denoising the corrupted image with PSNR= 29dB, ENL= 138, EPI= 0.20 using Edge Enhancing Diffusion method.EED model shows better performance compared to PMD model. Higher value of ENL refers to the effective reduction of speckle noise. EPI refers the preservation

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of edges in the restored image. From Table 2, it is clear that EED model shows better performance than PMD model with increase in PSNR, EPI and ENL values.

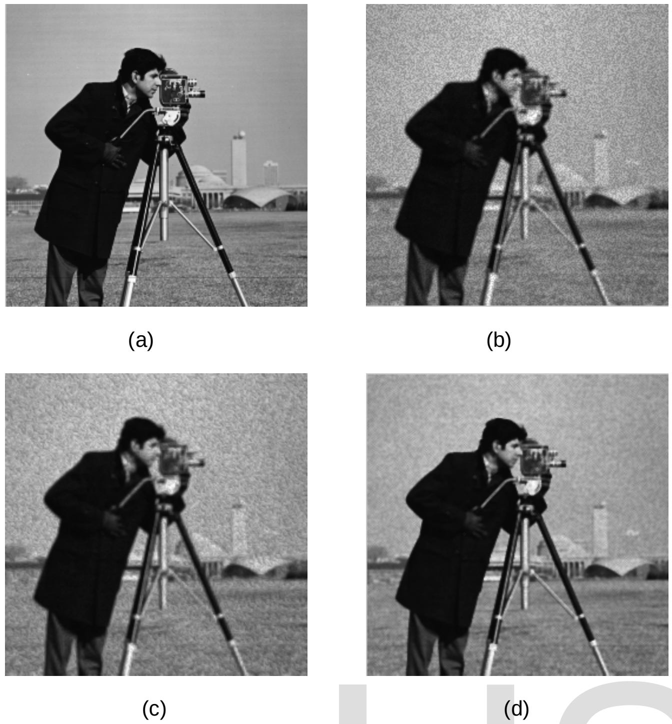

Fig. 6. (a) Original image, (b)Blurred noisy image, Restored images using (c) Perona Malik diffusion Model, (d) Edge enhancing diffusion Model.

4 Conclusion

In this paper, algorithms for image restoration have been presented. The Total variation and diffusion based methods are found to be effective in image denoising and deblurring. These methods are superior, since it forms one of the non blind deconvolution processes for image restoration. Simulation results show that the given algorithms can recover the original image which is corrupted by both additive and multiplicative noise.

References

[1] L.Rudin, S.Osher, E.Fatemi, “Nonlinear Total Variation based Noise Removal Algorithms”, *Physica D*, No.60, pp. 259–268, Nov. 1992.

[2] C.Boncelet, “Image Noise Models”, in Alan C. Bovik, *Handbook of Image and Video Processing*, Academic Press, July 2005.

[3] M.A.T. Figueiredo and J.M. Bioucas-Dias, “Restoration of Poissonian Images Using Alternating Direction Optimization”, IEEE Transactions on Image Processing, vol. 19, no. 12, pp. 3133-3145, Dec. 2010.

[4] Y.Zhu and Y.Shi, “A Fast Method for Reconstruction of Total-Variation MR Images With a Periodic Boundary Condition”, *IEEE Signal Processing Letters*, vol. 20, no. 4, pp. 291-294, April 2013.

[5] T.F. Chan and J.Shen, “Image Processing and Analysis: Variational, PDE, Wavelet, and Stochastic Methods”, *SIAM*, Sep. 2005.

[6] T.Pock, M.Pock and H.Bischof,”Algorithmic Differentiation: Application to Variational Problems in Computer Vision”, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 29, no. 7, pp. 1180-1193, July 2006.

[7] O.Scherzer, M.Grasmair, M.Haltmeier and F.Lenzen, “Variational Methods in Imaging”, *Applied Mathematical Sciences*, vol. 167, Oct. 2009.

[8] M.Elad, “Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing”, *Springer*, Sep. 2010.

[9] L.Fabbrini, M.Mario, M.Essina and G.Pinelli, “Improved Edge Enhancing Diffusion Filter for Speckle-Corrupted Images”, *IEEE Geoscience and Remote Sensing Letters*, vol. 11, no. 1, pp. 99-103, Jan. 2014.

[10] R.C. Gonzales and R.E Woods, “Digital Image Processing”, Second edition, *Prentice Hall*, Aug. 2002.

[11] M. A. T. Figueiredo, J. M. Bioucas-Dias, and R. D.Nowak, “Majorization–Minimization Algorithms for Wavelet-Based Image Restoration”, *IEEE Transactions on Image Processing*, vol. 16, no. 12, pp. 2980-2991, Dec. 2007.

[12] T.F. Chan, S.Osher, J.Shen, “ The Digital TV Filter and Nonlinear Denoising”, *IEEE Transactions on Image Processing*, vol. 10, no. 2, pp. 231-241, Feb. 2001.

[13] J.Weickert, “Anisotropic Diffusion in Image Processing”, B. G. Teubner, *Stuttgart*, Germany, Oct. 1998.

[14] T.Deschamps, M.Malladi and I.Ravve, “Fast Evolution of Image Manifolds and Application to Filtering and Segmentation in 3D Medical Images”, *IEEE Transactions on Visualization and Computer Graphics*, vol. 10, no. 5, pp. 525-535, Jan. 2004.

[15] P.Perona and J.Malik, “Scale-Space and Edge Detection Using Anisotropic Diffusion” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 12, no. 7, pp. 629-639, July 1990.